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On the Arithmetical Theory of Pencils of Binary Quadratic Forms.

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1. Given two binary quadratic forms:

$$A = ax^2 + 2bxy + cy^2,$$
$$B = a'x^2 + 2b'xy + c'y^2,$$

where the coefficients are integers, positive or negative or zero, the totality of forms obtained by giving to α and β all positive or negative or zero integer values in the expression $\alpha A + \beta B$ will be called a *pencil of forms* and will be denoted by (A, B). The forms A and B will be called the *base forms* of the pencil.

2. If $\overline{A} = \alpha A + \beta B$ and $\overline{B} = \gamma A + \delta B$ are two forms of the pencil, then the pencil $(\overline{A}, \overline{B})$ is said to be *contained* in the pencil (A, B). It is easily seen that any form that appears in $(\overline{A}, \overline{B})$ appears also in (A, B). The converse is not necessarily true. If two pencils are such that each is contained in the other, they are said to be *equivalent.*

3. THEOREM. *If two pencils are equivalent and their bases are related by the equations* $\overline{A} = \alpha A + \beta B$, $\overline{B} = \gamma A + \delta B$, *then* $\alpha\delta - \beta\gamma = \pm 1$.

For solving the two equations for A and B, we have

$$A = \frac{\delta}{\varepsilon}\overline{A} - \frac{\beta}{\varepsilon}\overline{B},$$

$$B = -\frac{\gamma}{\varepsilon}\overline{A} + \frac{\alpha}{\varepsilon}\overline{B},$$

where $\varepsilon = \alpha\delta - \beta\gamma$. It is necessary therefore that $\frac{\delta}{\varepsilon}$, $\frac{\beta}{\varepsilon}$, $\frac{\gamma}{\varepsilon}$ and $\frac{\alpha}{\varepsilon}$ be integers, and therefore that $\frac{\alpha\delta}{\varepsilon^2} - \frac{\beta\gamma}{\varepsilon^2}$ should be. But this expression is equal to $\frac{1}{\varepsilon}$, whence $\varepsilon = \pm 1$. The equivalence will be called proper or improper according as $\varepsilon = +1$ or $\varepsilon = -1$.

4. In the study of pencils two kinds of transformations will be used. The first, as above, connects the base forms of the two pencils. The pencil (A, B) will be said to be transformed into the pencil $(\overline{A}, \overline{B})$ by the *α-substitution* $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$. If, in the second place, the transformation $x = \lambda x' + \mu y'$, $y = \nu x' + \rho y'$ be applied to the *variables* in the two base forms A and B, two other forms are produced which may be taken as base forms of a new pencil $(\overline{A}', \overline{B}')$. The pencil will then be said to be transformed into the pencil $(\overline{A}', \overline{B}')$ by the *λ-substitution* $\begin{pmatrix}\lambda & \mu \\ \nu & \rho\end{pmatrix}$. It will be found that in this case the two pencils are not necessarily equivalent, even when $\lambda\rho - \mu\nu = \pm 1$. If $\lambda\rho - \mu\nu = \pm 1$, the pencils will be called *similar*, and the similarity will be called proper or improper according as $\lambda\rho - \mu\nu = +1$ or -1.

5. We shall denote by D and D' the determinants of A and B respectively, and the invariant $2bb' - ac' - a'c$ by F. The binary quadratic form $Dx^2 + Fxy + D'y^2$, which is of fundamental importance in the theory, we shall denote by H, and call the harmonic form of the pencil. The form $(ab' - a'b)x^2 + (ac' - a'c)xy + (bc' - b'c)y^2$, which except for a constant factor is the Jacobian of the two forms, will be denoted by J and called the Jacobian form of the pencil. The resultant of A and B, which is also the determinant of H and J, will be denoted by R. It has the value

$$R = F^2 - 4DD' = (ac' - a'c)^2 - 4(ab' - a'b)(bc' - b'c).$$

6. THEOREM. *Two equivalent pencils have the same resultant.*

If $\overline{A} = \alpha A + \beta B$, $\overline{B} = \gamma A + \delta B$, where $\alpha\delta - \beta\gamma = \pm 1$, we have

$$\overline{A} = (\alpha a + \beta a')x^2 + 2(\alpha b + \beta b')xy + (\alpha c + \beta c')y^2,$$
$$\overline{B} = (\gamma a + \delta a')x^2 + 2(\gamma b + \delta b')xy + (\gamma c + \delta c')y^2.$$

If now we compute the determinants $\overline{D}$ and $\overline{D}'$, of $\overline{A}$ and $\overline{B}$, we find:

$$\overline{D} = \alpha^2 D + \alpha\beta F + \beta^2 D',$$
$$\overline{D}' = \gamma^2 D + \gamma\delta F + \delta^2 D'.$$

Also

$$\overline{F} = 2\alpha\gamma D + F(\alpha\delta + \beta\gamma) + 2\beta\delta D'.$$

From these equations $\overline{R} = \overline{F}^2 - 4\overline{D}\overline{D}' = (\alpha\delta - \beta\gamma)^2 R = R$.

7. From the form of the expressions for $\overline{D}, \overline{D}'$ and $\overline{F}$ in the preceding paragraph, we infer immediately the following theorem:

THEOREM. *If the pencil $(\bar{A}, \bar{B})$ is contained in the pencil (A, B) and the bases are related by the equations $\bar{A} = \alpha A + \beta B$, $\bar{B} = \gamma A + \delta B$, then the form H of the pencil (A, B) goes into the form $\bar{H}$ of the pencil $(\bar{A}, \bar{B})$ by the substitution* $\begin{pmatrix}\alpha & \gamma \\ \beta & \delta\end{pmatrix}$.

8. As a particular case of the last theorem when $\alpha\delta - \beta\gamma = \pm 1$, we have the theorem:

THEOREM. *Two equivalent pencils have equivalent forms H, and the equivalence of the forms is proper or improper according as the equivalence of the pencils is proper or improper.*

9. THEOREM. *If J and $\bar{J}$ are the Jacobian forms of two equivalent pencils, then $J = \pm \bar{J}$, the upper sign or the lower occurring according as the equivalence of the pencils is proper or improper.*

Let the pencils be as given in paragraph 6. The form $\bar{J}$ belonging to the pencil $(\bar{A}, \bar{B})$ is found to be

$$\begin{aligned} &[(\alpha a + \beta a')(\gamma b + \delta b') - (\alpha b + \beta b')(\gamma a + \delta a')]\, x^2 \\ + &[(\alpha a + \beta a')(\gamma c + \delta c') - (\alpha c + \beta c')(\gamma a + \delta a')]\, xy \\ + &[(\alpha b + \beta b')(\gamma c + \delta c') - (\gamma b + \delta b')(\alpha c + \beta c')]\, y^2. \end{aligned}$$

This easily reduces to

$$(\alpha\delta - \beta\gamma)\,[(ab' - a'b)\,x^2 + (ac' - a'c)\,xy + (bc' - b'c)\,y^2],$$

whence $\bar{J} = (\alpha\delta - \beta\gamma)\,J$.

10. THEOREM. *Two similar pencils have the same forms H.*

Let the two pencils be (A, B) and $(\bar{A}', \bar{B}')$, where $A = (abc)$, $B = (a'b'c')$, $\bar{A}' = (\bar{a}\bar{b}\bar{c})$, $\bar{B}' = (\bar{a}'\bar{b}'\bar{c}')$ Then if the λ-substitution which connects the two pencils be $\begin{pmatrix}\lambda & \mu \\ \nu & \rho\end{pmatrix}$, we have the following equations:

$$\begin{aligned} \bar{a} &= a\lambda^2 + 2b\lambda\nu + c\nu^2, \\ \bar{b} &= a\lambda\mu + b(\lambda\rho + \mu\nu) + c\nu\rho, \\ \bar{c} &= a\mu^2 + 2b\mu\rho + c\rho^2, \\ \bar{a}' &= a'\lambda^2 + 2b'\lambda\nu + c'\nu^2, \\ \bar{b}' &= a'\lambda\mu + b'(\lambda\rho + \mu\nu) + c'\nu\rho, \\ \bar{c}' &= a'\mu^2 + 2b'\mu\rho + c'\rho^2. \end{aligned}$$

It is easily verified that

$$\bar{b}^2 - \bar{a}\bar{c} = (b^2 - ac)(\lambda\rho - \mu\nu)^2,$$
$$\bar{b}'^2 - \bar{a}'\bar{c}' = (b'^2 - a'c')(\lambda\rho - \mu\nu)^2,$$

and

$$2\bar{b}\bar{b}' - \bar{a}\bar{c}' - \bar{a}'\bar{c} = (2bb' - ac' - a'c)(\lambda\rho - \mu\nu)^2.$$

The H forms are thus identical, since $(\lambda\rho - \mu\nu)^2 = 1$.

11. THEOREM. *Two similar pencils have equivalent forms J; and the equivalence is proper or improper according as the similarity of the pencils is proper or improper.*

It is easily verified, using the notation of the preceding theorem, that

$$\bar{a}\bar{b}' - \bar{a}'\bar{b} = (ab' - a'b)\lambda^2 + (ac' - a'c)\lambda\nu + (bc' - b'c)\nu^2,$$
$$\bar{a}\bar{c}' - \bar{a}'\bar{c} = 2(ab' - a'b)\lambda\mu + (ac' - a'c)(\lambda\rho + \mu\nu) + 2(bc' - b'c)\nu\rho,$$
$$\bar{b}\bar{c}' - \bar{b}'\bar{c} = (ab' - a'b)\mu^2 - (ac' - a'c)\mu\rho + (bc' - b'c)\rho^2.$$

These are precisely the coefficients that would be obtained by transforming J by the substitution $\begin{pmatrix}\lambda & \mu \\ \nu & \rho\end{pmatrix}$.

12. THEOREM. *If two pencils are equivalent and also similar, then their λ-substitution is an automorph of J, and their α-substitution is an automorph of H.*

13. THEOREM. *Given a pencil, an equivalent pencil may always be found in which one of the base forms shall lack any desired coefficient.*

Let the pencil be (A, B), where $A = (abc)$ and $B = (a'b'c')$. If it is desired to find an equivalent pencil $(\bar{A}, \bar{B})$, such that the base form $\bar{A}$ shall lack the middle coefficient, choose α and β such that $\alpha b + \beta b' = 0$. This may be done in such a way that α and β shall be relatively prime. Thus if d is the greatest common factor of b and b', the values of α and β are b/d and b'/d. γ and δ may now be found such that $\alpha\delta - \beta\gamma = \pm 1$. Then the α-substitution $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ accomplishes the desired result. As γ and δ are not uniquely determined, an infinite number of such pencils may be found. It is not, however, in general possible to make more than one coefficient vanish. Thus if (γ, δ) are one pair of values such that $\alpha\delta - \beta\gamma = \pm 1$, then the other pairs are $\gamma + k\alpha$ and $\delta + k\beta$, where k is any integer. If now we wish to make the third coefficient, say, of $\bar{B}$, vanish, we must choose k so that

$$(\gamma + k\alpha)c + (\delta + k\beta)c' = 0,$$

whence

$$k = \frac{\gamma c + \delta c'}{\alpha c + \beta c'},$$

which is not generally integral.

14. PROBLEM. *To find a pencil that shall have a given Jacobian form.*

We suppose, without loss of generality by the preceding theorem, that $b = 0$. We have then,

$$\begin{array}{llll} (1) & -ac = D, & (4) & ab' = L, \\ (2) & b'^2 - a'c' = D', & (5) & ac' - a'c = M, \\ (3) & -ac' - a'c = F, & (6) & -b'c = N; \end{array}$$

where L, M and N are the coefficients of the given form J (note that M is the entire coefficient of the term xy, not $2M$). From equations (4) and (6) we have $a/c = -L/M$, and this with (1) gives $a^2 = DL/N$ and $c^2 = DN/L$. From (3) and (5) we have $ac' = (M - F)/2$ and $a'c = +(M + F)/2$. Let now d be the greatest common divisor of L and N, so that $L = dL'$ and $N = dN'$, where L' and N' are relatively prime. Since $a^2 = DL/N = DL'/N'$ and $c^2 = DN'/L'$, it is clear that D must be a multiple of L' and of N' and that the multiple must be a square. Put then $D = k^2L'N'$, and we get:

$$\begin{aligned} a &= kL', \\ c &= -kN', \\ a' &= (M + F)/2kN', \\ b' &= d/k, \\ c' &= (M - F)/2kL'. \end{aligned}$$

From these equations it appears that $M + F$ and $M - F$ are divisible by $2k$ and that therefore k divides both M and F. k also divides d, and since d divides L and M, k is a factor of $R = M^2 - 4LN$.

It also appears that

$$F \equiv M (\mathrm{mod}\ 2kL')$$

and

$$F \equiv -M (\mathrm{mod}\ 2kN'),$$

so that the values of F which are available differ by multiples of $2kL'N'$ or $2D/k$.

The form H which goes with the above pencil is

$$H = k^2L'N'x^2 + Fxy + [d^2/k^2 + (F^2 - M^2)/4k^2N'L']\, y^2.$$

If now we take another F, say $F + 2Dg/k$, where g is any integer, we get for the corresponding form H'

$$H' = k^2N'L'x^2 + (F + 2Dg/k)xy + [d^2/k^2 + (\overline{F + 2Dg/k}^{2} - M^2)/4kLN]\, y^2.$$

It is easily verified that H' may be obtained from H by the substitution $\begin{pmatrix} 1 & \frac{g}{k} \\ 0 & 1 \end{pmatrix}$, so that if g/k is an integer H and H' are properly equivalent, also that the two pencils that correspond to these two values of F are connected by the α-substitution $\begin{pmatrix} 1 & 0 \\ \frac{g}{k} & 1 \end{pmatrix}$. It follows readily that if two values of g differ by a multiple of k, the corresponding pencils are equivalent, and that therefore for any value of k which is a divisor of d there are k classes of non-equivalent pencils having the same Jacobian form J. It is easily seen also that two pencils obtained from the distinct positive values of k are not equivalent. Indeed one goes into the other by the α-substitution $\begin{pmatrix} \frac{k'}{k} & 0 \\ 0 & \frac{k}{k'} \end{pmatrix}$, which is not integral in the coefficients unless $k = \pm k'$. It follows therefore that:

The number of non-equivalent pencils having a given Jacobian-form is $\Sigma_d k$, *where* k *runs through all of the positive divisors of* d, d *being the greatest common divisor of* L *and* N.

In particular, if J is primitive, $k = 1$ and there is only one class of pencils having J for a Jacobian form. From the above solution of the problem it also appears that:

15. THEOREM. *Two pencils having the same Jacobian form are equivalent, if their harmonic forms are equivalent, and if the Jacobian form is primitive the two pencils are always equivalent.*

16. As an illustration of the solution of the problem of the preceding paragraphs let it be required to find all pencils having the Jacobian form

$$J = 10x^2 + 20xy + 100y^2.$$

Here $d = 10$, $L' = 1$, $N' = 10$, $k = 1, 2, 5$ and 10. The total number of non-equivalent classes of pencils will therefore be $1 + 2 + 5 + 10 = 18$.

The general base forms for the pencils will be

$$A = (k,\ 0,\ -10k), \qquad B = (\overline{20 - F}/2k,\ 10/k,\ \overline{20 - F}/2k).$$

When $k = 1$, $F \equiv 0 \pmod{20}$, giving the class represented by the pencil

$$A = (1,\ 0,\ -10), \qquad B = (1,\ 10,\ 10).$$

When $k = 2$, we have the two classes represented by the pencils:

$$A = (2,\ 0,\ 20), \qquad B = (1,\ 5,\ 0), \qquad \text{where } F \equiv 20 \pmod{80};$$
$$A = (2,\ 0,\ -20), \qquad B = (2,\ 5,\ -10), \qquad \text{where } F \equiv 60 \pmod{80}.$$

When k equals 5, we have the five classes corresponding to the values $F \equiv 80, 180, 280, 380, 480 \pmod{500}$, and represented by the pencils:

$(5, 0, -50)$,	$(1, 2, -6)$,
$(5, 0, -50)$,	$(2, 2, -16)$,
$(5, 0, -50)$,	$(3, 2, -26)$,
$(5, 0, -50)$,	$(4, 2, -36)$,
$(5, 0, -50)$,	$(5, 2, -46)$.

When k equals 10, we have ten classes corresponding to the values $F \equiv 180, 380, 580, 780, 980, 1180, 1380, 1580, 1780, 1980 \pmod{2000}$, and represented by the pencils $A = (10, 0, -100)$ and B the following forms:

$(1, 1, -8)$, $(2, 1, -18)$, $(3, 1, -28)$, $(4, 1, -38)$, $(5, 1, -48)$,
$(6, 1, -58)$, $(7, 1, -68)$, $(8, 1, -78)$, $(9, 1, -88)$, $(10, 1, -98)$.

17. THEOREM. *The harmonic form is the duplicate of the Jacobian form.*

This theorem follows from the identity:

$$\begin{aligned}&[(ab' - a'b)x^2 + (ac' - a'c)xy + (bc' - b'c)y^2]\,[(ab' - a'b)x'^2 + (ac' - a'c)x'y' \\ &\quad + (bc' - b'c)y'^2] = (b^2 - ac)\,[a'xx' + b'(xy' + x'y) + c'yy']^2 \\ &\quad - (2bb' - ac' - a'c)\,.\,[a'xx' + b'(xy' + x'y) + c'yy']\,[axx' + b(xy' + x'y) \\ &\quad + cyy'] + (b'^2 - a'c')\,[axx' + b(xy' + x'y) + cyy']^2.\end{aligned}$$

This indicates that if in the form H we make the bilinear substitution,

$$\begin{aligned}X &= a'xx' + b'xy' + b'x'y + c'yy', \\ Y &= -axx' - bxy' - bx'y - cyy',\end{aligned}$$

we get $J(xy)\,.\,J(x'y')$ as above.

18. THEOREM. *If the Jacobian form is primitive, then the harmonic form belongs to the principal genus.*

This follows from the preceding theorem by well-known principles in the composition of classes (cf. Mathews, "Theory of Numbers," p. 171).

19. THEOREM. *If H belongs to the principal genus, it will serve as the harmonic form of some pencil.*

For by a theorem, due to Gauss, every such form is obtainable as the duplicate of some form J. The pencil with this form J for its Jacobian form will have the given form H for a harmonic form.

20. THEOREM. *If the form M is contained in a pencil, the determinant of the form is representable by the harmonic form of pencil.*

For if $M = \alpha A + \beta B$, the determinant of M is found (see paragraph 6) to be

$$D\alpha^2 + F\alpha\beta + D'\beta^2.$$

21. From the preceding theorems, it appears that the determinant of every form contained in a pencil is a quadratic residue of the resultant R of the pencil, and that conversely the resultant R is a quadratic residue of the determinant of every form contained in the pencil. Further, since a quadratic residue of a number is a quadratic residue of every divisor of that number, it follows that every factor of R is a residue of every factor of the determinant of every form contained in the pencil; and conversely, every factor of the determinant of any form contained in the pencil is a residue of every factor of R, the pencil always being supposed primitive. As a special theorem, if R is divisible by a prime of the form $4n - 1$, then every odd prime divisor of the determinant of any form contained in the pencil must be of the form $4n + 1$. For by the law of quadratic reciprocity there are no two primes both of the form $4n - 1$ each of which is a quadratic residue of the other.

22. THEOREM. *If a form $M = (l, m, n)$ is contained in a pencil $A = (a, b, c)$, $B = (a', b', c')$, then $(ab' - a'b)(bn - cm) = (bc' - b'c)(am - bl)$, and conversely.*

For if $M = \alpha A + \beta B$, then

$$l = \alpha a + \beta a',$$
$$m = \alpha b + \beta b',$$
$$n = \alpha c + \beta c';$$

and putting these values in the equation which is to be verified, we have the identity:

$$(ab' - a'b)(\alpha bc + \beta bc' - \alpha cb - \beta b'c)$$
$$= (bc' - b'c)(\alpha ab + \beta ab' - \alpha ab - \beta a'b).$$

Starting from this identity the converse is easily established.

23. THEOREM. *Given any two pencils, there is one form primitively represented by both.*[*] *If there is more than one, then one form is contained in the other and there are an infinite number of such forms.*

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[*] A form $M = \alpha A + \beta B$ is *primitively* represented by the pencil (A, B) when the parameters α and β are prime to each other.

Let the two pencils be (A, B) and (M, N), where $A = (a, b, c)$, $B = (a', b', c')$, $M = (l, m, n)$, $N = (l', m', n')$. From the equation $\alpha A + \beta B = \gamma M + \delta N$ we have

$$\begin{aligned} \alpha a + \beta a' &= \gamma l + \delta l', \\ \alpha b + \beta b' &= \gamma m + \delta m', \\ \alpha c + \beta c' &= \gamma n + \delta n'. \end{aligned}$$

Eliminating α, we have

$$\begin{aligned} \beta (a b' - a' b) &= \gamma (a m - b l) + \delta (a m' - b l'), \\ \beta (b c' - b' c) &= \gamma (b n - c m) + \delta (b n' - c m'). \end{aligned}$$

Eliminating β, we have

$$\begin{aligned} \gamma \, &[(a b' - a' b)(b n - c m) - (b c' - b' c)(a m - b l)] \\ &+ \delta \, [(a b' - a' b)(b n' - c m') - (b c' - b' c)(a m' - b l')] = 0. \end{aligned}$$

If now the forms M and N are not contained in the pencil (A, B), then the coefficients of γ and δ are not zero, by the preceding theorem. Dividing out extraneous factors, we have, for determining γ and δ, an equation $P\gamma + Q\delta = 0$, where P and Q are relatively prime to each other. From this $\gamma = kQ$ and $\delta = -kP$, where k is an arbitrary integer, which may be so chosen as to make β and α integral. Different values of k will lead to forms which are identical except for a constant factor.

The theorem still holds when one of the forms, M say, is contained in the pencil (A, B). In that case the coefficient of γ vanishes. Thus $\delta =$ zero and α, β, γ are to be obtained from the equations

$$\begin{aligned} \alpha a + \beta a' &= \gamma l, \\ \alpha b + \beta b' &= \gamma m, \\ \alpha c + \beta c' &= \gamma n, \end{aligned}$$

whence eliminating α, we have

$$\begin{aligned} \beta (a b' - a' b) &= \gamma (a m - b l), \\ \beta (b c' - b' c) &= \gamma (b n - c m), \\ \beta (a c' - a' c) &= \gamma (a n - c l). \end{aligned}$$

These equations will serve to determine β and γ uniquely, and so also α uniquely except for a constant factor, unless the coefficients of β and γ all vanish, in which case the pencil (A, B) is of a trivial sort, the base forms being identical except for a constant factor. The values of α, β and γ manifestly give the representation of M by the pencil (A, B).

In case both forms belong to the pencil (A, B), it is clear that the pencil (M, N) is contained in the pencil (A, B); and so in this case there are an infinite number of forms primitively representable by both pencils.

24. Applying the above theorem to two similar pencils which are related by the λ-transformation $\begin{pmatrix}\lambda & \mu \\ \nu & \rho\end{pmatrix}$, we have the theorem:

THEOREM. *There are two and only two forms primitively representable by a pencil which are equivalent under a given transformation.*

For let ϕ be that one form primitively representable by the pencil P and by the pencil P' similar to it. Now every form of P is equivalent to some form of P' by the transformation $\begin{pmatrix}\lambda & \mu \\ \nu & \rho\end{pmatrix}$. Some form of P, f say, is therefore equivalent to ϕ by this transformation.

This theorem does not imply that every form of a pencil is equivalent to one and only one other form in that pencil, but that a given transformation reveals one and only one such pair of equivalent forms.

25. An exception to the above theorem arises when the λ-transformation is an automorph of J (see paragraph 19). In that case there are an infinite number of forms primitively representable by the pencil which are equivalent under the transformation. In fact every form of the pencil has a form in the pencil equivalent to it for every automorphic transformation of J. If now R is positive, the number of automorphic transformations of J is infinite, and the number of forms in the pencil equivalent to any given form in the pencil is also infinite. If, however, the number of automorphs of J is finite, as in the case when R is negative, then the number of forms in a pencil which are equivalent to any given form of the pencil is finite. In any case the number is not greater than the number of representations of the determinant of the given form by the harmonic form of the pencil.